In all of the following questions, assume that \( \vec{x}, \vec{y} \) and \( \vec{z} \) are the standard orthonomal basis vectors. It is okay to write down your answer as a multiplication of multiple matrices.

**Basic linear algebra**

1. Write down the matrix form of the linear transformation that maps the vector \( \vec{x} = (1, 0, 0) \) to \( \vec{x}' = (1, 3, 2) \), \( \vec{y} = (0, 1, 0) \) to \( \vec{y}' = (3, 2, 1) \), and \( \vec{z} = (0, 0, 1) \) to \( \vec{z}' = (1, 4, 2) \).

2. Write down the matrix that changes the coordinates of a vector under the standard basis \( \vec{x}, \vec{y}, \vec{z} \), to the coordinates under the new basis \( \vec{x}' = (4, 5, 0), \vec{y}' = (-3, 4, 0), \vec{z}' = (0, 0, 1) \).

   (a) Now consider the vector \( \vec{v} \) whose coordinates are \( (3, 2, 1) \) under \( \vec{x}, \vec{y}, \vec{z} \). What are its coordinates under the new basis \( \vec{x}', \vec{y}', \vec{z}' \)?

3. Prove that the multiplication of two orthonormal matrices \( Q_1 \) and \( Q_2 \) is still orthonormal. That is, prove that \( Q_1 Q_2 \) is orthonormal if \( Q_1 \) and \( Q_2 \) are orthonormal.

**Eigenvalues and eigenvectors**

In all of the following questions, limit your attention only to the real positive eigenvalues and their associated eigenvectors.

1. Consider the \( 3 \times 3 \) matrix \( T \) that rotates the input vector by 30 degree counter-clockwise along the \( z \)-axis. Write down all real eigenvalue and eigenvector pairs of \( T \).

2. Consider a \( 3 \times 3 \) symmetric matrix \( T \) whose eigenvalues are \( 8, 5, 2 \). Given a unit vector \( \vec{v} \) what are the minimum and maximum lengths of \( T\vec{v} \)?

3. Does an \( n \times n \) matrix always have \( n \) eigenvalues?

4. Consider an orthonormal matrix \( T \). What is the maximum number of distinct real positive eigenvalues of \( T \)?

5. Write down the matrix \( T \) that stretches the input vector by 3 along \( \left( \frac{4}{5}, \frac{3}{5}, 0 \right) \), by 4 along \( \left( -\frac{3}{5}, \frac{4}{5}, 0 \right) \) and by 2 along \( (0, 0, 1) \).

   (a) How many unique eigenvalues does it have?

   (b) What are the smallest eigenvalue and the corresponding eigenvector of \( T \)?
Singular value decomposition

1. Consider the linear transformation \( T \) that does the following:
\( T \) first stretches the input vector by 3 along \((\frac{4}{5}, \frac{3}{5}, 0)\), by 4 along \((-\frac{3}{5}, \frac{4}{5}, 0)\) and by 2 along \((0, 0, 1)\). It then rotates the stretched vector by 90 degree counter-clockwise along the z-axis.

(a) Write down the matrix representation of \( T \) under the standard basis \( \vec{x}, \vec{y}, \vec{z} \).
(b) Write down the singular value decomposition of \( T \).

2. Consider matrix \( T \) whose singular value decomposition is as follows:
\[
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
5 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

(a) Describe the transformation represented by \( T \) in terms of rotations and stretching. To get the full credit, your description should be concise.
(b) Write down all real positive eigenvalues of \( T \) and their corresponding eigenvectors.
(c) Write down all eigenvalue and eigenvector pairs of \( T^T T \).